

The Reeh–Schlieder Property for Ground States

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Abstract. Recently it has been shown that the Reeh–Schlieder property w.r.t. thermal equilibrium states is a direct consequence of locality, additivity and the relativistic KMS condition. Here we extend this result to ground states.

1. Introduction

The general theory of quantized fields is mostly concerned with the vacuum sector, since this is the appropriate framework for traditional (i.e., few particle) high energy physics. However, speculations about a phase-transition (at high temperatures and high densities) from standard hadronic matter to a state which is commonly called the quark–gluon plasma activated some interest in thermal sectors. Recent experiments on cold Bose and Fermi gases have renewed the interest in ground states or (low temperature) thermal states. Just as in the vacuum sector we suppose that some of the most peculiar predictions of the theory will be traced back to the famous Reeh–Schlieder property [RS], although this may take some time and effort. For instance, the significance of the Reeh–Schlieder property for quantum information theory was realized only recently (see, e.g., [CH][SW]).) Despite its direct physical significance, the Reeh–Schlieder property will turn out be indispensable as a technical tool, e.g., for scattering or superselection theory in the new sectors.

According to the standard arguments the Reeh–Schlieder property is a property of finite energy states in the vacuum sector. One might argue that all physically relevant states should be locally normal w.r.t. the vacuum representation and therefore, whenever the Reeh–Schlieder property is urgently needed, one may take recourse to the vacuum sector. But obviously it is more enlightening to prove the Reeh–Schlieder property directly in the sector that is under investigation (see, e.g., [Jä][Ju] for KMS states). Moreover, it is not always possible to take recourse to the vacuum sector. In lower space–time dimensions KMS states, which fail to be locally normal, exist (see, e.g., the discussion in [BJu]). Despite the general belief that this can not happen in 3+1 space–time dimensions, all attempts to rule out infrared problems, which may destroy local normality, failed up to now.

From a technical viewpoint we would like to emphasize that the original arguments of Reeh and Schlieder [RS] were based on the global symmetry properties of the vacuum state. Thus the real challenge may come from quantum field theory on curved space–times (see [St] and [V] for free fields) where the curved background will, at least in general, not allow global symmetries. (See however, [BoB] and [BEM] for highly symmetric space–times.) Thus the question arises whether one can abandon the assumption that the translations are

unitarily implemented. And indeed, it has been shown that the Reeh–Schlieder property w.r.t. thermal equilibrium states is a direct consequence of locality, additivity and the relativistic KMS condition of Bros and Buchholz [BB] which does not exclude KMS states braking the rotation or translation symmetry [Jä]. In this letter we extend this result to a class of ground states (in Minkowski space, of course), which satisfy a similar ‘relativistic ground state condition’. Once again the essential steps of our proof are based on a theorem of Glaser. They exploit only the characteristic analyticity properties of a relativistic ground state; whether or not the translation or/and rotation symmetries are spontaneously broken turns out to be irrelevant for the Reeh–Schlieder property. From a technical viewpoint the original contribution of this letter is contained in Lemma 3.4.

To conclude this introduction, we briefly outline the content of this letter. In Section 2 we introduce a (relativistic) ground state condition and discuss some aspects of the corresponding GNS representations. Section 3 contains the derivation of the Reeh–Schlieder property for ground states.

2. Relativistic Ground States

In the algebraic formulation [H] a QFT is casted into an inclusion preserving map

$$\mathcal{O} \rightarrow \mathcal{A}(\mathcal{O}) \tag{1}$$

which assigns to any open bounded region \mathcal{O} in Minkowski space \mathbb{R}^4 a unital C^* -algebra $\mathcal{A}(\mathcal{O})$. The Hermitian elements of the *abstract* C^* -algebra $\mathcal{A}(\mathcal{O})$ are interpreted as the observables which can be measured at times and locations in \mathcal{O} . The net $\mathcal{O} \rightarrow \mathcal{A}(\mathcal{O})$ is isotonus, i.e., there exists a unital embedding

$$\mathcal{A}(\mathcal{O}_1) \hookrightarrow \mathcal{A}(\mathcal{O}_2) \quad \text{if } \mathcal{O}_1 \subset \mathcal{O}_2. \tag{2}$$

For mathematical convenience the local algebras are embedded in the C^* -inductive limit algebra

$$\mathcal{A} = \overline{\bigcup_{\mathcal{O} \subset \mathbb{R}^4} \mathcal{A}(\mathcal{O})}^{C^*}. \tag{3}$$

The space–time symmetry of Minkowski space manifests itself in the existence of a representation

$$\alpha: (\Lambda, x) \mapsto \alpha_{\Lambda, x} \in \text{Aut}(\mathcal{A}), \quad (\Lambda, x) \in \mathcal{P}_+^\uparrow, \tag{4}$$

of the (orthochronous) Poincaré group \mathcal{P}_+^\uparrow . Lorentz transformations Λ and space–time translations x act geometrically:

$$\alpha_{\Lambda, x}(\mathcal{A}(\mathcal{O})) = \mathcal{A}(\Lambda\mathcal{O} + x) \quad \forall (\Lambda, x) \in \mathcal{P}_+^\uparrow. \tag{5}$$

Without loss of generality, we may assume that the space–time translations $\alpha: \mathbb{R}^4 \rightarrow \text{Aut}(\mathcal{A})$ are strongly continuous, i.e., for each $a \in \mathcal{A}$

$$\|\alpha_x(a) - a\| \rightarrow 0 \quad \text{as } x \rightarrow 0. \tag{6}$$

Observables localized in spacelike separated space–time regions commute, i.e.,

$$\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}^c(\mathcal{O}_2) \quad \text{if} \quad \mathcal{O}_1 \subset \mathcal{O}_2'. \quad (7)$$

Here \mathcal{O}' denotes the spacelike complement of \mathcal{O} and $\mathcal{A}^c(\mathcal{O})$ denotes the set of operators in \mathcal{A} which commute with all operators in $\mathcal{A}(\mathcal{O})$.

States are, by definition, positive, linear and normalized functionals over \mathcal{A} . If stable crystals exist in a relativistic framework, then they will certainly break the spatial translation and rotation symmetry of space and time. Consequently, one should not (and we do not) require that these symmetries can be unitarily implemented in the GNS representation associated with such a state. The maximal propagation velocity of signals, however, will not be affected by such a lack of symmetry. It is simply characteristic for any relativistic theory. Following Bros and Buchholz [BB] we propose that it manifests itself in the following ‘relativistic ground state condition’.

Definition 2.1. *A time invariant state ω_∞ is a relativistic ground state if and only if for every pair of elements a, b of \mathcal{A} there exists a function $\mathcal{F}_{a,b}$ which is bounded and analytic in a convex open tube*

$$-\mathcal{T} \times \mathcal{T}, \quad \mathcal{T} \subset \{z \in \mathbb{C} : \Im z \in V_+\}. \quad (8)$$

where the basis \mathcal{C} of $\mathcal{T} = \mathbb{R}^4 + i\mathcal{C}$ is a neighbourhood in \mathbb{R}^4 of the linear segment

$$\{y \in \mathbb{R}^4 : y = \lambda e, \lambda > 0\}, \quad (9)$$

e is a timelike unit vector and at all boundary points x in \mathbb{R}^4 the cone Λ_x with apex in x , which is the union of all closed half-lines starting from x and intersecting \mathcal{C} , is the light cone $V_+ = \{y \in \mathbb{R}^4 : y^0 > |\vec{y}|\}$. Moreover, $\mathcal{F}_{a,b}$ is continuous at the boundary set $\mathbb{R}^4 \times \mathbb{R}^4$ with boundary values given by

$$\mathcal{F}_{a,b}(x_1, x_2) = \omega_\infty(\alpha_{x_1}(a)\alpha_{x_2}(b)) \quad \forall x_1, x_2 \in \mathbb{R}^4. \quad (10)$$

Obviously (see condition (ii) in the following theorem), relativistic ground states are ground states in the usual sense. Recall [BR, 5.3.19] the following

Theorem 2.2. *A state ω_∞ is called a ground state if it satisfies one (and thus all) of the following four equivalent conditions w.r.t. some unit vector e in the forward light-cone V_+ :*

- (i) *If $a, b \in \mathcal{A}_{\alpha_e}$, then the entire analytic function*

$$z \mapsto \omega_\infty(a\alpha_{\tau e}(b)) \quad (11)$$

is uniformly bounded in the region $\{\tau \in \mathbb{C} : \Im \tau \geq 0\}$. $\mathcal{A}_{\alpha_e} \subset \mathcal{A}$ denotes the set of analytic elements for the one-parameter subgroup $t \mapsto \alpha_{te}$.

- (ii) For any $a, b \in \mathcal{A}$ there exists a function $F_{a,b}$ which is continuous in $\Im z \geq 0$ and analytic and bounded in $\Im z > 0$. Moreover,

$$F_{a,b}(t) = \omega_\infty(\alpha_{-te/2}(a)\alpha_{te/2}(b)) \quad \forall t \in \mathbb{R}. \quad (12)$$

- (iii) Let \mathcal{D} denote the set of infinitely differentiable functions with compact support. If f is a function with Fourier transform $\tilde{f} \in \mathcal{D}$ and $\text{supp } \tilde{f} \subset (-\infty, 0)$, then

$$\omega_\infty(\alpha_f(a)^*\alpha_f(a)) = 0 \quad \forall a \in \mathcal{A}. \quad (13)$$

Here $\alpha_f(a) := \int dt f(t)\alpha_{te}(a)$.

- (iv) ω_∞ is time invariant, and if, in the GNS representation $(\pi_\infty, \mathcal{H}_\infty, \Omega_\infty)$,

$$e^{itH_\infty}\pi_\infty(a)\Omega_\infty = \pi_\infty(\alpha_{te}(a))\Omega_\infty \quad (14)$$

is the corresponding unitary representation of the time evolution $t \mapsto \alpha_{te}$ on \mathcal{H}_∞ , then

$$H_\infty \geq 0. \quad (15)$$

If these conditions are satisfied, then $U(t) := e^{itH_\infty} \in \pi_\infty(\mathcal{A})''$ for all $t \in \mathbb{R}$. Note that there might be a distinguished time direction e ; if there is none (as it is the case for the vacuum), then any timelike unit vector can be used.

Obviously, the set of ground states K_∞ is a weak*-closed convex subset of the state space. We recall that the decomposition of a ground state into extremal ground states is in general not unique. (Note that the KMS states for a fixed temperature form a simplex, thus the decomposition of KMS states into extremal ones is always unique.) But the decomposition of ground states shows another simple geometric property not generally shared by the set of KMS states. The ground states form a face, i.e., if a ground state

$$\omega_\infty = \sum_{i=1}^n \lambda_i \omega_i \quad (16)$$

is a finite convex combination of arbitrary states then each ω_i , $i = 1, \dots, n$, is automatically a ground state, too.

If $\omega \in K_\infty$ is an extremal ground state, then ω is pure, i.e.,

$$\pi_\omega(\mathcal{A})'' = \mathcal{B}(\mathcal{H}_\omega) \quad \text{and} \quad \pi_\omega(\mathcal{A})' = \mathbb{C} \cdot \mathbf{1}. \quad (17)$$

If the pair (\mathcal{A}, ω) is \mathbb{R} -abelian, i.e.,

$$\inf_{a' \in C_0(\alpha_{\mathbb{R}e}(a))} |\omega'([a', b])| = 0 \quad (18)$$

for all $a, b \in \mathcal{A}$ and all time invariant vector states ω' of π_ω , then $\pi_\omega(\mathcal{A})'$ is (at least) abelian. Here $C_o(\alpha_{\mathbb{R}e}(a))$ denotes the convex hull of $\{\alpha_{te}(a) : t \in \mathbb{R}\}$.

Let us recall some more well known properties (taken from [BR, 5.3.40]):

Theorem 2.3. *Let K_∞ be the set of ground states. The following statements are equivalent:*

- (i) *The pair (\mathcal{A}, ω) is \mathbb{R} -abelian for all $\omega \in K_\infty$.*
- (ii) *$\pi_\omega(\mathcal{A})'$ is abelian for all $\omega \in K_\infty$.*
- (iii) *K_∞ is a simplex; i.e., there exists a unique decomposition into extremal ground states.*
- (iv) *Each pure ground state is weakly clustering in the sense that*

$$\inf_{a' \in C_o(\alpha_{\mathbb{R}e}(a))} |\omega(a'b) - \omega(a)\omega(b)| = 0 \quad (19)$$

for all $a, b \in \mathcal{A}$.

- (v) *If $\omega \in K_\infty$ is a factor state, then ω is pure.*
- (vi) *If $\omega_1, \omega_2 \in K_\infty$ are pure states, then ω_1 and ω_2 are either disjoint or equal.*
- (vii) *If ω_1 and ω_2 are distinct pure states in K_∞ , then the face generated by ω_1 and ω_2 in the set of time invariant states is equal to the convex set*

$$\left\{ \lambda\omega_1 + (1 - \lambda)\omega_2 : \lambda \in [0, 1] \right\}. \quad (20)$$

Let us now return to relativistic ground states. The GNS representation π_∞ assigns to any $\mathcal{O} \subset \mathbb{R}^4$ a von Neumann algebra

$$\mathcal{R}_\infty(\mathcal{O}) = \pi_\infty(\mathcal{A}(\mathcal{O}))''. \quad (21)$$

Definition 2.4. *The net $\mathcal{O} \rightarrow \mathcal{R}_\infty(\mathcal{O})$ is called additive, if*

$$\cup_{i \in I} \mathcal{O}_i = \mathcal{O} \Rightarrow \vee_{i \in I} \mathcal{R}_\infty(\mathcal{O}_i) = \mathcal{R}_\infty(\mathcal{O}). \quad (22)$$

Here I is some index set and $\vee_i \mathcal{R}_\infty(\mathcal{O}_i)$ denotes the von Neumann algebra generated by the algebras $\mathcal{R}_\infty(\mathcal{O}_i)$, $i \in I$.

Remark. If ω_∞ is locally normal w.r.t. the vacuum representation, then additivity in the vacuum sector and additivity in the ground state sector are equivalent. As is well known, additivity in the vacuum sector can be proven, if the net of local algebras is constructed from a Wightman field theory.

3. The Reeh–Schlieder Property

We start with the following

Proposition 3.1. *Let ω_∞ be a state which satisfies the relativistic ground state condition and let \mathcal{V} be an open neighborhood of the origin in \mathbb{R}^4 . It follows that for each complex-valued test function f with support in \mathcal{V}*

$$\int_{\mathbb{R}^4 \times \mathbb{R}^4} d^4 y_1 d^4 y_2 \mathcal{F}_{a^*, a}(y_1 - i\kappa e, y_2 + i\kappa e) \overline{f(y_1)} f(y_2) \geq 0 \quad (23)$$

for all $\kappa > 0$. Here e denotes the unit vector introduced in Theorem 2.2, which might not be unique.

Proof. Let $a \in \mathcal{A}_\alpha$ be an entire analytic element for the translations. Put

$$\Psi_f := \int_{\mathcal{V}} d^4 y_1 f(y_1) \alpha_{y_1}(\alpha_{i\kappa e}(a)) \Omega_\infty \in \mathcal{H}_\infty. \quad (24)$$

Exploring the definition (10) of $\mathcal{F}_{a^*, a}$ one finds

$$\int_{\mathbb{R}^4 \times \mathbb{R}^4} d^4 y_1 d^4 y_2 \mathcal{F}_{a^*, a}(y_1 - i\kappa e, y_2 + i\kappa e) \overline{f(y_1)} f(y_2) = \|\Psi_f\|^2 \geq 0. \quad (25)$$

For general $a \in \mathcal{A}$, choose a sequence $\{a_n \in \mathcal{A}_\alpha\}_{n \in \mathbb{N}}$ such that

$$\|a_n\| \leq \|a\| \quad \text{and} \quad \pi_\infty(a_n) \Omega_\infty \rightarrow \pi_\infty(a) \Omega_\infty \quad \text{as } n \rightarrow \infty. \quad (26)$$

Now define, for $y_1, y_2 \in \mathbb{R}^4$ and $\kappa > 0$,

$$\mathcal{F}_n(y_1 - i\kappa e, y_2 + i\kappa e) := \mathcal{F}_{a_n^*, a_n}(y_1 - i\kappa e, y_2 + i\kappa e). \quad (27)$$

The maximum modulus principle [R] implies that

$$\left| \mathcal{F}_n(y_1 - i\kappa e, y_2 + i\kappa e) - \mathcal{F}_m(y_1 - i\kappa e, y_2 + i\kappa e) \right| \quad (28)$$

assumes its maximum value on the boundary of its domain and for $\kappa = 0$, the boundary value, the relativistic ground state condition yields

$$\begin{aligned} & \lim_{\kappa \searrow 0} \left| \mathcal{F}_n(y_1 - i\kappa e, y_2 + i\kappa e) - \mathcal{F}_m(y_1 - i\kappa e, y_2 + i\kappa e) \right| \\ & \leq \sup_{y_1, y_2 \in \mathbb{R}^4} \left| \omega_\infty(\alpha_{y_1}(a_n^*) \alpha_{y_2}(a_n)) - \omega_\infty(\alpha_{y_1}(a_n^*) \alpha_{y_2}(a_m)) \right| \\ & \quad + \sup_{y_1, y_2 \in \mathbb{R}^4} \left| \omega_\infty(\alpha_{y_1}(a_n^*) \alpha_{y_2}(a_m)) - \omega_\infty(\alpha_{y_1}(a_m^*) \alpha_{y_2}(a_m)) \right| \\ & \leq \|a\| \sup_{y_2 \in \mathbb{R}^4} \left\| \pi_\infty(\alpha_{y_2}(a_n - a_m)) \right\| + \|a\| \sup_{y_1 \in \mathbb{R}^4} \left\| \pi_\infty(\alpha_{y_1}(a_m^* - a_n^*)) \right\|. \quad (29) \end{aligned}$$

In the last inequality we have used $\|a_n\| = \|a_n^*\| \leq \|a\|$ and $\|\Omega_\infty\| = 1$. Continuity of the translations (recall that all automorphisms of a C^* -algebra are continuous), i.e.,

$$\lim_{a_m \rightarrow a_n} \|\pi_\infty(\alpha_y(a_m - a_n))\| = 0, \quad (30)$$

now implies that $\{\mathcal{F}_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence uniformly on $\overline{\mathcal{U}}$, where

$$\mathcal{U} := \{(y_1 - i\kappa e, y_2 + i\kappa e) : y_1, y_2 \in \mathbb{R}^4, \kappa > 0\}. \quad (31)$$

The limit function \mathcal{F}_∞ is therefore continuous and bounded on $\overline{\mathcal{U}}$ and analytic in \mathcal{U} . By construction,

$$\mathcal{F}_\infty(y_1, y_2) = \mathcal{F}_{a^*, a}(y_1, y_2) \quad \forall y_1, y_2 \in \mathbb{R}^4. \quad (32)$$

Thus, due to their analyticity properties, the functions \mathcal{F}_∞ and $\mathcal{F}_{a^*, a}$ must coincide on \mathcal{U} . It follows that

$$\begin{aligned} \int_{\mathbb{R}^4 \times \mathbb{R}^4} d^4 y_1 d^4 y_2 \mathcal{F}_{a^*, a}(y_1 - i\kappa e, y_2 + i\kappa e) \overline{f(y_1)} f(y_2) = \\ \lim_{n \rightarrow \infty} \int_{\mathbb{R}^4 \times \mathbb{R}^4} d^4 y_1 d^4 y_2 \mathcal{F}_n(y_1 - i\kappa e, y_2 + i\kappa e) \overline{f(y_1)} f(y_2) \geq 0. \end{aligned} \quad (33)$$

□

The next step uses an adapted and simplified version of Glaser's Theorem 1 ([G a], see also [G b][BEM]):

Theorem 3.2. (Glaser): *Let $a \in \mathcal{A}$ and let $\mathcal{F}_{a^*, a}$ denote the function introduced in (10). The following properties are equivalent:*

- i.) *There exists an open neighborhood \mathcal{V} of 0 in \mathbb{R}^4 and a point $z_1 \in \mathcal{T}$ such that $z_1 + \mathcal{V} \subset \mathcal{T}$ and such that for each complex-valued testfunction f with support in \mathcal{V}*

$$\int_{\mathbb{R}^4 \times \mathbb{R}^4} d^4 y_1 d^4 y_2 \mathcal{F}_{a^*, a}(y_1 + \bar{z}_1, y_2 + z_1) \overline{f(y_1)} f(y_2) \geq 0. \quad (34)$$

- ii.) *There exists a sequence $\{f_a^{(n)} : \mathcal{T} \rightarrow \mathbb{C}\}_{n \in \mathbb{N}}$ of functions holomorphic in \mathcal{T} such that for $(z_1, z_2) \in -\mathcal{T} \times \mathcal{T}$*

$$\mathcal{F}_{a^*, a}(z_1, z_2) = \sum_{n=1}^{\infty} \overline{f_a^{(n)}(\bar{z}_1)} f_a^{(n)}(z_2) \quad (35)$$

holds in the sense of uniform convergence on every compact subset of $-\mathcal{T} \times \mathcal{T}$.

The crucial step in the proof of the Reeh–Schlieder property is now summarized in the following

Theorem 3.3. *For each $a \in \mathcal{A}$ the vector valued function $\Phi_a: \mathbb{R}^4 \rightarrow \mathcal{H}_\infty$,*

$$x \mapsto \pi_\infty(\alpha_x(a))\Omega_\infty \quad (36)$$

can be analytically continued from the real axis into the tube \mathcal{T} such that it is weakly continuous for $\Im z \searrow 0$.

Since Ω_∞ need not be separating for $\pi_\infty(\mathcal{A})''$, the map $b \mapsto \pi_\infty(b)\Omega_\infty$ will in general *not* be injective. Hence we can not immediately apply the arguments used in the case of KMS states. However, the following lemma assures that at least the map $\pi_\infty(b)\Omega_\infty \mapsto \mathcal{F}_{b^*,b}(x_1, x_2)$ is well-defined.

Lemma 3.4. *Assume $\pi_\infty(b)\Omega_\infty = 0$. It follows that*

$$\mathcal{F}_{b^*,b}(z_1, z_2) = 0 \quad \forall (z_1, z_2) \in -\mathcal{T} \times \mathcal{T}. \quad (37)$$

Proof. Since a ground state is required to be time invariant, the time evolution can always be unitarily implemented in the GNS representation (see (14)):

$$U(t)\pi_\infty(b)\Omega_\infty := \pi_\infty(\alpha_{te}(b))\Omega_\infty. \quad (38)$$

Thus $U(t)\Omega_\infty = \Omega_\infty$ for all $t \in \mathbb{R}$ and therefore $\pi_\infty(b)\Omega_\infty = 0$ implies

$$\pi_\infty(\alpha_{te}(b))\Omega_\infty = U(t)\pi_\infty(b)U(-t)\Omega_\infty = 0 \quad \forall t \in \mathbb{R}. \quad (39)$$

Consequently, see (12),

$$F_{b^*,b}(t + i\kappa) = 0 \quad \forall t \in \mathbb{R}, \quad \forall \kappa > 0. \quad (40)$$

Now let e denote the distinguished unit vector in the time direction. (If there is a distinguished time direction; otherwise e can be any timelike unit vector.) It follows that

$$\begin{aligned} \mathcal{F}_{b^*,b}\left(-\frac{1}{2}(t + i\kappa)e, \frac{1}{2}(t + i\kappa)e\right) &= F_{b^*,b}(t + i\kappa) \\ &= 0 \quad \forall t \in \mathbb{R}, \quad \forall \kappa > 0. \end{aligned} \quad (41)$$

Moreover, for $z_1 \in \mathcal{T}$,

$$\begin{aligned} \mathcal{F}_{b^*,b}(\bar{z}_1, z_1) &= \sum_{n=1}^{\infty} \overline{f_b^{(n)}(z_1)} f_b^{(n)}(z_1) \\ &= \sum_{n=1}^{\infty} |f_b^{(n)}(z_1)|^2 \end{aligned} \quad (42)$$

is a positive bounded function. Therefore it takes its minimum on the boundary of \mathcal{T} ; but according to equation (41) $F_{b^*,b}(\bar{z}_1, z_1)$ takes the minimal possible value, namely zero, at interior points (e.g., $F_{b^*,b}(-i\kappa e, i\kappa e) = 0$ for $\kappa > 0$). Consequently, $F_{b^*,b}(\bar{z}_1, z_1)$ has to vanish identically. Inspecting the r.h.s. of (42) we find that

$$f_b^{(n)}(z_1) = 0 \quad \forall z_1 \in \mathcal{T} \quad \forall n \in \mathbb{N}. \quad (43)$$

Consequently,

$$\mathcal{F}_{b^*,b}(z_1, z_2) = 0 \quad \forall (z_1, z_2) \in -\mathcal{T} \times \mathcal{T}. \quad (44)$$

□

Proof. (of Theorem 3.3). Let $a, b \in \mathcal{A}$ with $\|a\| = 1$. According to Proposition 3.1 and Theorem 3.2 there exists a sequence $\{f_a^{(n)}: \mathcal{T} \rightarrow \mathbb{C}\}_{n \in \mathbb{N}}$ of functions holomorphic in \mathcal{T} which satisfies (35). Lemma 3.4 now allows us to consider — for $z \in \mathcal{T}$ and $a \in \mathcal{A}$ fixed — the map $\hat{\phi}_{a,z}: \mathcal{S} \rightarrow \mathbb{C}$

$$\pi_\infty(b)\Omega_\infty \mapsto \sum_{n=1}^{\infty} \overline{f_a^{(n)}(z)} f_b^{(n)}(0). \quad (45)$$

Here \mathcal{S} denotes the set of vectors $\mathcal{S} := \{\pi_\infty(b)\Omega_\beta : b \in \mathcal{A}\}$. Now $\overline{\pi_\infty(\mathcal{A})\Omega_\infty} = \mathcal{H}_\infty$ and

$$\left| \sum_{n \in \mathbb{N}} \overline{f_a^{(n)}(z)} f_b^{(n)}(0) \right|^2 \leq F_{a^*,a}(\bar{z}, z) \cdot \|\pi_\infty(b)\Omega_\infty\|^2. \quad (46)$$

Thus the Hahn–Banach Theorem allows us to extend the map $\hat{\phi}_{a,z}: \mathcal{S} \rightarrow \mathbb{C}$ to a (bounded) continuous linear functional $\phi_{a,z}$ on \mathcal{H}_∞ . The Riesz Lemma ensures that there exists a vector $\Phi_a(z) \in \mathcal{H}_\infty$ such that

$$\phi_{a,z}(\Psi) = (\Phi_a(z), \Psi) \quad \forall \Psi \in \mathcal{H}_\infty. \quad (47)$$

As can easily be seen (by choosing once again an appropriate sequence $\{a_n\}_{n \in \mathbb{N}}$ of analytic elements), the map

$$z \mapsto \Phi_a(z) \quad (48)$$

is analytic for $z \in \mathcal{T}$ and weakly continuous at the boundary set $\Im z = 0$, where it satisfies

$$\Phi_a(x) = \pi_\infty(\alpha_x(a))\Omega_\infty \quad \forall x \in \mathbb{R}^4. \quad (49)$$

□

Without proof we mention the following

Corollary 3.5. *Let $a, b \in \mathcal{A}$ and let Φ_a, Φ_b denote the associated vector valued functions introduced in (48). It follows that*

$$\mathcal{F}_{a^*,b}(\bar{z}_1, z_2) = (\Phi_a(z_1), \Phi_b(z_2)) \quad (50)$$

for all $z_1, z_2 \in \mathcal{T}$. Here $\mathcal{F}_{a^*,b}$ denotes the analytic function introduced in (10).

What remains to be proven in order to establish the Reeh–Schlieder property is fairly standard. Borchers and Buchholz [BoB] recently gave a nice and transparent formulation of this part of the argument and since it has already been reproduce in [Jä], we feel free to refer the reader to the literature cited. Thus we simply state our result.

Theorem 3.6. *Consider a QFT as specified in Section 2 and let ω_∞ be a state, which satisfies the relativistic ground state condition. If the additivity assumption (22) holds, then*

$$\mathcal{H}_\infty = \overline{\pi_\infty(\mathcal{A}(\mathcal{O}))\Omega_\infty}, \quad (51)$$

for any open space–time region $\mathcal{O} \subset \mathbb{R}^4$. Moreover, if the spacelike complement of \mathcal{O} is not empty, then Ω_∞ is separating for $\mathcal{R}_\infty(\mathcal{O})$.

Similar to the situation in the vacuum and the KMS sector, Ω_∞ shares the Reeh–Schlieder property with a large class of vectors in \mathcal{H}_∞ .

Corollary 3.7. *Consider a QFT as specified in Section 2 and let ω_∞ be a state, which satisfies the relativistic ground state condition. Moreover, assume the additivity assumption (22) holds. It follows that there exists a dense set $\mathcal{D}_\alpha \subset \mathcal{H}_\infty$, namely*

$$\mathcal{D}_\alpha = \left\{ \left(\mathbb{1} - \frac{\pi_\infty(a)}{2\|a\|} \right) \Omega_\infty : a \in \mathcal{A}_\alpha \right\}, \quad (52)$$

such that for all $\Psi \in \mathcal{D}_\alpha$

$$\mathcal{H}_\infty = \overline{\pi_\infty(\mathcal{A}(\mathcal{O}))\Psi}, \quad (53)$$

where $\mathcal{O} \subset \mathbb{R}^4$ is again an arbitrary open space–time region.

Remark. The essential step is to show that for arbitrary $b \in \mathcal{A}$ the function

$$\mathbb{R}^4 \ni x \mapsto \pi_\infty(\alpha_x(b))\Psi \quad (54)$$

extends to some analytic vector-valued function in the domain \mathcal{T} . The reader is invited to check that Proposition 3.1 and Theorem 3.3 as well as Lemma 3.4 can easily be adapted and that the proofs given remain valid if we replace Ω_∞ by some vector $\Psi \in \mathcal{D}_\alpha$.

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